BAULKHAM HILLS

2023 **HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION**

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 70

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 60 marks (pages 6 – 10)

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10.

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1 Which of the following is an expression for $\int \sin^2 2x\,dx$?

A. $\frac{x}{2}-\frac{1}{8}\sin 4x+C$

B. $\frac{x}{2}+\frac{1}{8}\sin 4x+C$

C. $\frac{x}{2}-\frac{1}{4}\sin 4x+C$

D. $\frac{x}{2}+\frac{1}{4}\sin 4x+C$

2 Which of the following Cartesian equations represents the parametric equations $x=3\sin\theta+1$ and $y=4\cos\theta-2$?

A. $\frac{(x-1)^2}{3}+\frac{(y+2)^2}{4}=1$

B. $\frac{(x+1)^2}{3}+\frac{(y-2)^2}{4}=1$

C. $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{16}=1$

D. $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{16}=1$

3 Which vector is perpendicular to $\underset{\sim}{v}=2\underset{\sim}{i}-5\underset{\sim}{j}$?

A. $\underset{\sim}{a}=-5\underset{\sim}{i}+2\underset{\sim}{j}$

B. $\underset{\sim}{b}=5\underset{\sim}{i}-2\underset{\sim}{j}$

C. $\underset{\sim}{c}=2\underset{\sim}{i}+5\underset{\sim}{j}$

D. $\underset{\sim}{d}=5\underset{\sim}{i}+2\underset{\sim}{j}$

4 For which of the following differential equations is $y = 2e^{2x}$ a solution?

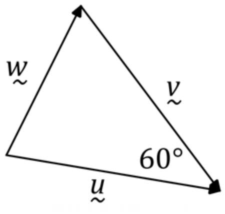
A. $\dfrac{d^2y}{dx^2} - 4y = 8e^{2x}$

B. $\dfrac{d^2y}{dx^2} - 4y = e^{2x}$

C. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 0$

D. $\dfrac{d^2y}{dx^2} - 2y\dfrac{dy}{dx} = 0$

5 Vectors $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are shown below.



Which of the following statements is true?

A. $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2$

B. $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u}\underset{\sim}{v}|$

C. $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 + |\underset{\sim}{u}\underset{\sim}{v}|$

D. $|\underset{\sim}{w}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - |\underset{\sim}{u}||\underset{\sim}{v}|$

6 A bag contains 20 balls such that 10 balls are red, 7 are white and 3 are blue. What is the minimum of balls that must be picked up from the bag blindfolded to be assured of picking at least one ball of each colour?

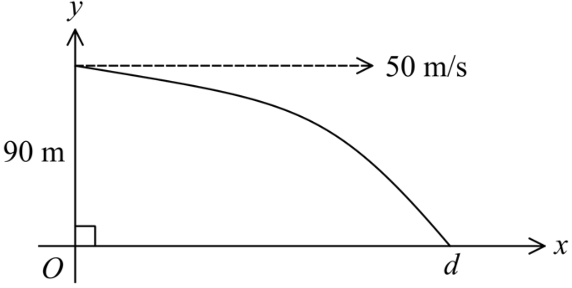
A. 11

B. 12

C. 14

D. 18

7 The diagram below shows the trajectory of a ball thrown horizontally, at a speed of 50 ms^{-1}, from the top of a tower 90 metres above ground level.



After what time does the ball strike the ground?

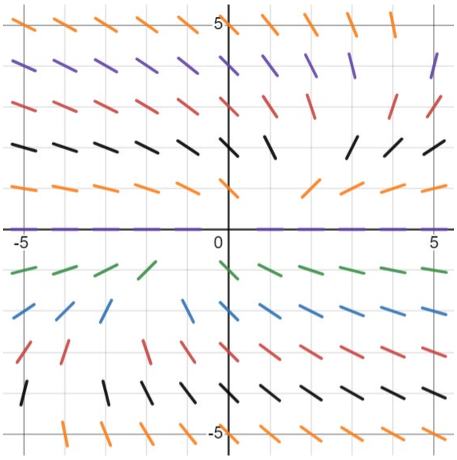
A. $3\sqrt{\frac{5}{g}}$ seconds

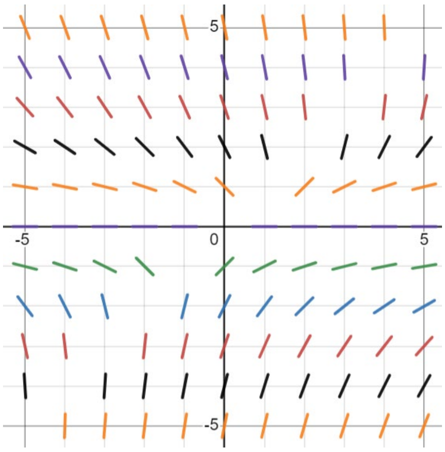
B. $3\sqrt{\frac{50}{g}}$ seconds

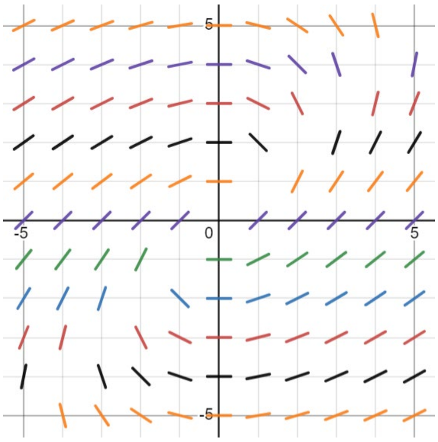
C. $6\sqrt{\frac{5}{g}}$ seconds

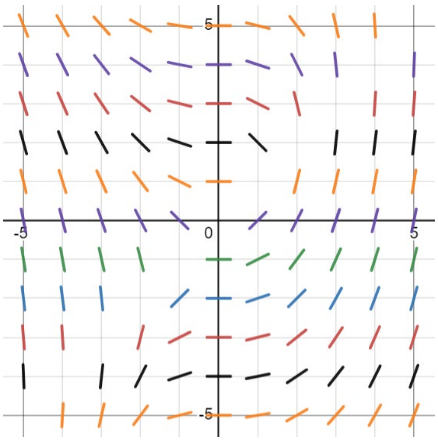
D. $6\sqrt{\frac{50}{g}}$ seconds

8 Which of the following best represents the direction field for the differential equation $\frac{dy}{dx} = \frac{y^2}{x - y}$?

A.


B.


C.


D.


9 What is the derivative of $x\cos^{-1}\left(x\right)-\sqrt{1-x^2}$?

A. $\dfrac{-1}{\sqrt{1-x^2}}$

B. $\dfrac{-2}{\sqrt{1-x^2}}$

C. $\cos^{-1}x$

D. $\sin^{-1}x$

10 What is the value of $(f^{-1})'(3)$ given $f(x)=x^3-3x^2+3x+1$?

A. $\dfrac{1}{12}$

B. $\dfrac{1}{3}$

C. 3

D. Undefined

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

In Questions 11 – 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Evaluate $\int_0^3 \frac{dx}{\sqrt{36-x^2}}$. **2**

(b) At a dinner party, the host, hostess and their 6 guests sit at a round table. In how many ways can they be arranged if:

(i) the host and hostess are separated? **2**

(ii) the host and hostess are sitting directly opposite each other? **1**

(c) Two fair six-sided dice are rolled together and the sum of the numbers on their upper faces is recorded. The result is recorded as either a 7 or not a 7.

(i) If p is the probability of a sum of 7, find the value of p. **1**

(ii) The two dice are rolled 30 times. Find the mean and variance of this binomial distribution. **2**

(d) Consider the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$.

(i) Find the unit vector $\hat{\underset{\sim}{a}}$. **1**

(ii) Find the scalar projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$. **2**

(e) (i) Prove $\cot\theta - 2\cot 2\theta = \tan\theta$. **2**

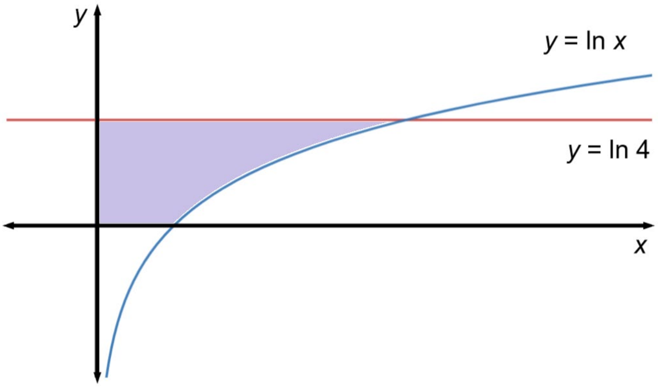
(ii) Hence, or otherwise, prove $\tan\theta + 2\tan 2\theta + 4\tan 4\theta = \cot\theta - 8\cot 8\theta$. **2**

End of Question 11

Question 12 (15 marks) Use the Question 12 Writing Booklet.

(a) Find $\int \sin 8x \cos 5x \, dx$. **2**

(b) In the diagram below, the shaded region bounded by the curve $y = \ln x$, x-axis, y-axis and the line $y = \ln 4$, is rotated about the y-axis. **3**

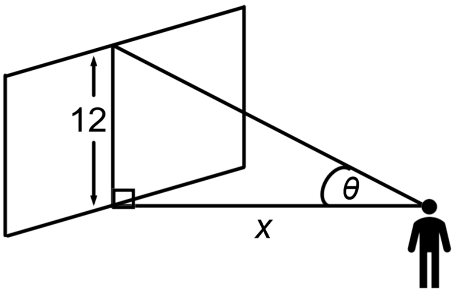


Find the exact volume of the solid of revolution.

(c) (i) Express $6\cos x + 8\sin x$ in the form $A\cos(x-\alpha)$, where $A > 0$ and $0 \leq \alpha \leq \frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve the equation $6\cos x + 8\sin x = 5$ for $0 \leq x \leq 2\pi$. Write your answers correct to three decimal places. **2**

(d) The top of a large theatre screen is 12 metres above Tom's eye level. Tom walks away from the screen at a rate of 1.1 metres/second while looking at the screen. Let the angle of elevation to the top of the screen from Tom's eye level be θ and the distance perpendicular to the screen be x, as shown below. **3**



Find the rate of change of the angle of elevation when Tom is 10 metres away from the screen, correct to three decimal places.

(e) Let the polynomial $P(x) = (x+1)(x-3)Q(x) + R(x)$, where $Q(x)$ and $R(x)$ are polynomials. When $P(x)$ is divided by $(x+1)$ the remainder is -7. When $P(x)$ is divided by $(x-3)$ the remainder is 1. Find $R(x)$. **3**

End of Question 12

Question 13 (15 marks) Use the Question 13 Writing Booklet.

(a) Use the substitution $u = 2x + 1$ to evaluate $\int_0^4 \frac{x}{\sqrt{2x+1}} dx$. **3**

(b) Use mathematical induction to prove that: **3**

$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + ... + \frac{1}{n \times (n+1)} = \frac{n}{n+1}$ for all positive integers *n*.

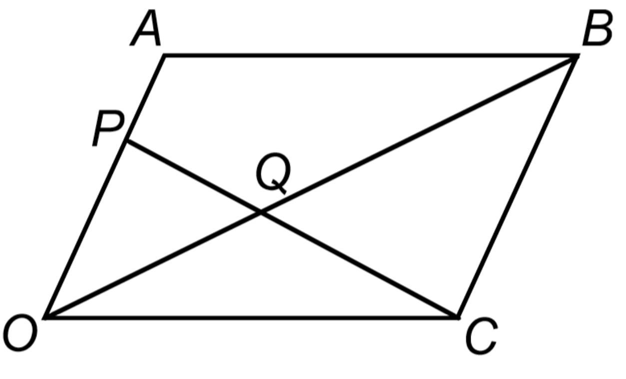
(c) A test consists of five multiple choice questions. Each question has four alternative answers. For each question only one of the alternative answers is correct. Jerry randomly selects an answer to each of the five questions.

(i) What is the probability that Jerry selects three correct and two incorrect answers? **2**

(ii) What is the probability that Jerry selects at least one incorrect answer? **1**

(d) A laptop's battery is considered faulty if its battery life is less than 2 hours. **3**
The laptop supplier knows that the chance of a faulty battery in any laptop is 7%.
A random sample of 50 laptops is selected from the supplier and the battery life of each laptop is tested.
Assuming the sample proportion is normally distributed, what is the probability that the percentage of laptops with faulty batteries lies between 5% and 10%?
Give your answer to the nearest percentage.

(e) In the diagram below, *OABC* is a parallelogram. Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OC} = \underset{\sim}{c}$, *P* be a point on *OA* so that $OP : PA = 2 : 1$ and *Q* be the intersection of *PC* and *OB* as shown. **3**



Given that $\overrightarrow{OQ} = \lambda \overrightarrow{OB}$, where λ is a constant, use vector methods to find the value of λ.

End of Question 13

Question 14 (15 marks) Use the Question 14 Writing Booklet.

(a) Consider the function $f(x) = \sin^{-1}\left(\frac{x}{2}\right)$.

(i) State the domain and range of $f(x)$. **2**

(ii) Determine whether $f(x)$ is odd, even or neither. Justify your answer. **1**

(iii) Hence, or otherwise, sketch the curve $y = \frac{1}{f(x)}$, without using calculus. **2**

(b) In the town of Eulerville, the population was estimated in the year 2000 to be 50 000. The population growth is given by the logistic equation $\frac{dP}{dt} = 0.05P\left(\frac{C-P}{C}\right)$, where t is the number of years after 2000 and C is the carrying capacity. **4**

In the year 2023, the population was estimated to be 80 000.
Use the fact that $\frac{C}{P(C-P)} = \frac{1}{P} + \frac{1}{C-P}$ to find the carrying capacity, to the nearest thousand.

Question 14 continues on page 10

(c) A ball is thrown from the origin, O, with a velocity, V m/s, at an angle of elevation of θ, where $\theta \neq 90^{\circ}$. You may assume that the position vector $\underset{\sim}{r}(t)$ of the ball in metres, from O, after t seconds is given by:

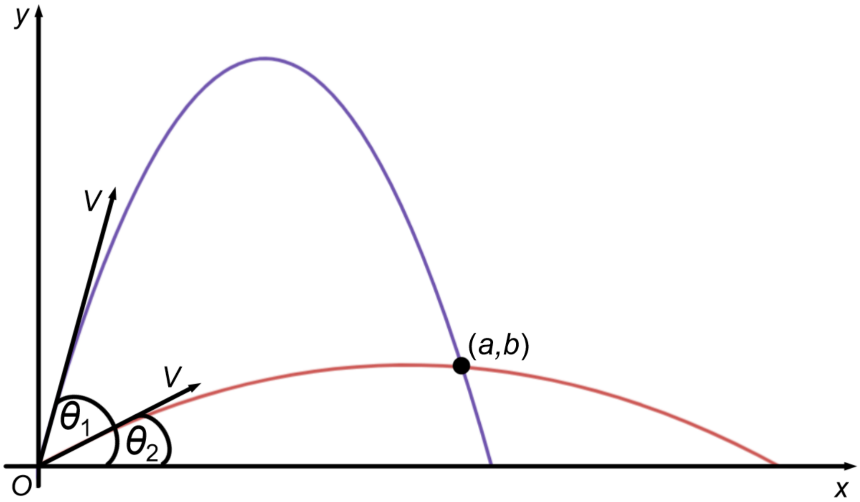
$$\underset{\sim}{r}(t) = Vt\cos\theta\underset{\sim}{i} + \left(Vt\sin\theta - \frac{1}{2}gt^2\right)\underset{\sim}{j} \qquad \text{(Do not prove this)}$$

where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the unit vectors in the horizontal and vertical directions respectively.

(i) Show that the equation of flight of the ball is: **2**

$$y = x\tan\theta - \frac{x^2}{4h}(1+\tan^2\theta)\text{, where } h = \frac{V^2}{2g}.$$

(ii) Two balls are thrown simultaneously with the same velocity, V m/s, at angles of elevation of θ_1 and θ_2 as shown below. **2**



Show that in order for the two balls to pass through the point (a,b):

$$a^2 < 4h(h-b).$$

(iii) Show that if two balls are thrown simultaneously with the same velocity, V m/s, and at angles of elevation of θ_1 and θ_2, such that $\theta_1 < 45^{\circ}$ and $\theta_2 < 45^{\circ}$, then the two balls will **not** pass through the point (a,b). **2**

End of paper

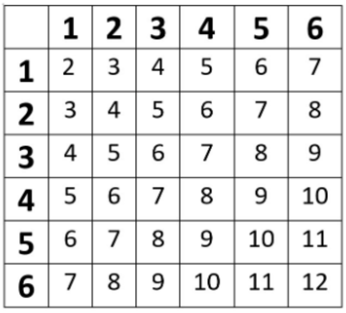
BAULKHAM HILLS HIGH SCHOOL

2023 YEAR 12 EXTENSION 1 TRIAL HSC SOLUTIONS

Suggested Solutions		Marks
Section I		
1. A	$\int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)dx$ $= \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + C$ $= \frac{x}{2} - \frac{1}{8}\sin 4x + C$	**1**
2. C	$x = 3\sin\theta + 1,\ y = 4\cos\theta - 2$ $\frac{x-1}{3} = \sin\theta \qquad (1)$ $\frac{y+2}{4} = \cos\theta \qquad (2)$ $(1)^2 + (2)^2:$ $\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+2}{4}\right)^2 = 1$ $\frac{(x-1)^2}{9} + \frac{(y+2)^2}{16} = 1$	**1**
3.D	$\begin{pmatrix}2\\-5\end{pmatrix} \bullet \begin{pmatrix}a\\b\end{pmatrix} = 0$ $2a - 5b = 0$ $a = \frac{5}{2}b$ $\therefore \underset{\sim}{d} = 5\underset{\sim}{i} + 2\underset{\sim}{j}$ is perpendicular to $\underset{\sim}{v}$.	**1**
4. C	$y = 2e^{2x}$ $\frac{dy}{dx} = 4e^{2x}$ $\frac{d^2y}{dx^2} = 8e^{2x}$ $(\text{C}): \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 0$ $\text{LHS } = 8e^{2x} - 2\left(4e^{2x}\right)$ $= 0$ $= \text{RHS}$	**1**

5. D	$\underset{\sim}{w}+\underset{\sim}{v}=\underset{\sim}{u}$ $\underset{\sim}{w}=\underset{\sim}{u}-\underset{\sim}{v}$ $\underset{\sim}{w}\cdot\underset{\sim}{w}=(\underset{\sim}{u}-\underset{\sim}{v})\cdot(\underset{\sim}{u}-\underset{\sim}{v})$ $\quad=\underset{\sim}{u}^2-2\underset{\sim}{u}\underset{\sim}{v}+\underset{\sim}{v}^2$ $\lvert\underset{\sim}{w}\rvert^2=\lvert\underset{\sim}{u}\rvert^2+\lvert\underset{\sim}{v}\rvert^2-2\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert\cos 60$ $\therefore\lvert\underset{\sim}{w}\rvert^2=\lvert\underset{\sim}{u}\rvert^2+\lvert\underset{\sim}{v}\rvert^2-\lvert\underset{\sim}{u}\rvert\lvert\underset{\sim}{v}\rvert$	**1**
6. D	You are able to pick 17 balls (10 red and 7 white), before you picking a blue. Hence the minimum number of balls to ensure at least one ball of each colour is 18.	**1**
7. C	$\ddot{y}=-g$ $\dot{y}=-gt$ $y=-\frac{1}{2}gt^2$ When $y=-90$: $-\frac{1}{2}gt^2=-90$ $t^2=\frac{180}{g}$ $t=\sqrt{\frac{180}{g}}$ $=6\sqrt{\frac{5}{g}}$	**1**
8. B	$y=0$ is a constant solution, eliminating C and D. Testing a point (1, -1): $\frac{dy}{dx}=\frac{(-1)^2}{1-(-1)}$ $=\frac{1}{2}$ $\therefore$ The answer is B.	**1**
9. C	$\frac{d}{dx}\left(x\cos^{-1}(x)-\sqrt{1-x^2}\right)$ $=x\times\frac{-1}{\sqrt{1-x^2}}+\cos^{-1}x-\frac{1}{2\sqrt{1-x^2}}\times -2x$ $=\cos^{-1}x$	**1**

10. B	$f(x)=x^3-3x^2+3x+1$ When $f(x)$ = 3: $3=x^3-3x^2+3x+1$ $1=x^3-3x^2+3x-1$ $1=\left(x-1\right)^3$ $\therefore x=2$ $f(x)$ passes through the point (2, 3) $(f^{-1})'(3)=\dfrac{1}{f'(b)}$, where $b=2$ $(f^{-1})'(3)=\dfrac{1}{3x^2-6x+3}$ $(f^{-1})'(3)=\dfrac{1}{3(2)^2-6(2)+3}$ $=\dfrac{1}{3}$	**1**

<table>
<tr><th>Section II</th><th>Suggested Solutions</th><th>Mk</th><th>Comments</th></tr>
<tr><td>11(a)</td><td>$\int_0^3 \frac{dx}{\sqrt{36-x^2}}=\left[\sin^{-1}\frac{x}{6}\right]_0^3$
$=\sin^{-1}\left(\frac{1}{2}\right)-0$
$=\frac{\pi}{6}$</td><td>2</td><td>2Mk: Provides correct solution.
1Mk: Correctly integrates expression.</td></tr>
<tr><td>11(b)(i)</td><td>Total – sitting together:
$7!-2\times 6!$
$=3600$</td><td>2</td><td>2Mk: Provides correct solution.
1Mk: Obtains sitting together correctly or equivalent.</td></tr>
<tr><td>11(b)(ii)</td><td>6! = 720</td><td>1</td><td>1Mk: Provides correct solution.</td></tr>
<tr><td>11(c)(i)</td><td>There are 6 ways to obtain a sum of 7.

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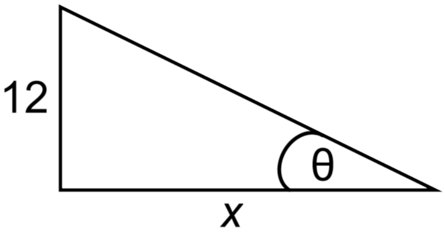
<tr><td>6</td><td>7</td><td>8</td><td>9</td><td>10</td><td>11</td><td>12</td></tr>
</table>
Probability of obtaining a 7 is $\frac{1}{6}$.
i.e. $p=\frac{1}{6}$.</td><td>1</td><td>1Mk: Provides correct solution.</td></tr>
</table>

11(c)(ii)	$E(X)=np$ $=30\times\frac{1}{6}$ $=5$ $Var(X)=np(1-p)$ $=30\times\frac{1}{6}\times\left(1-\frac{1}{6}\right)$ $=\frac{25}{6}$	**2**	2Mk: Provides both *E(X)* and *Var(X)*. 1Mk: Obtains *E(X)* or *Var(X)*.
11(d)(i)	$\left\lvert\hat{\underset{\sim}{a}}\right\rvert=\sqrt{4^2+(-3)^2}$ $=5$ $\hat{\underset{\sim}{a}}=\frac{1}{5}(4\underset{\sim}{i}-3\underset{\sim}{j})$ $=\frac{4}{5}\underset{\sim}{i}-\frac{3}{5}\underset{\sim}{j}$	**1**	1Mk: Provides correct solution.
11(d)(ii)	$\left\lvert\text{proj}_{\underset{\sim}{a}}\underset{\sim}{b}\right\rvert=\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}$ $=\left(3\underset{\sim}{i}+2\underset{\sim}{j}\right)\cdot\left(\frac{4}{5}\underset{\sim}{i}-\frac{3}{5}\underset{\sim}{j}\right)$ $=3\times\frac{4}{5}+2\times\left(-\frac{3}{5}\right)$ $=\frac{6}{5}$	**2**	2Mk: Provides correct solution. 1Mk: Obtains a correct expression to find the scalar projection or finds the vector projection.
11(e)(i)	$\text{LHS}=\cot\theta-2\cot2\theta$ $=\frac{1}{\tan\theta}-2\times\frac{1}{\tan2\theta}$ $=\frac{1}{\tan\theta}-2\times\frac{1-\tan^2\theta}{2\tan\theta}$ $=\frac{1}{\tan\theta}-\frac{1}{\tan\theta}+\tan\theta$ $=\tan\theta$ $=\text{RHS}$	**2**	2Mk: Provides correct proof. 1Mk: Correctly expands a double angle.
11(e)(ii)	$\tan\theta=\cot\theta-2\cot2\theta$ from part (i) $\tan2\theta=\cot2\theta-2\cot4\theta$ $\tan4\theta=\cot4\theta-2\cot8\theta$ Substitute these results into the identity. $\text{LHS}=\tan\theta+2\tan2\theta+4\tan4\theta$ $=(\cot\theta-2\cot2\theta)+2(\cot2\theta-2\cot4\theta)+4(\cot4\theta$ $-2\cot8\theta)$ $=\cot\theta-8\cot8\theta$ $=\text{RHS}$	**2**	2Mk: Provides correct proof 1Mk: Obtains $\tan2\theta$ or $\tan4\theta$ in terms of cot.

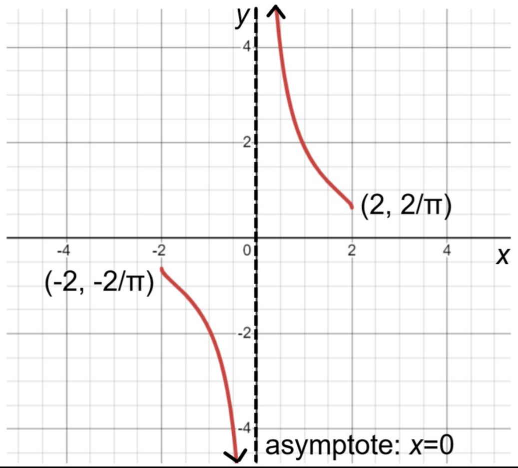
12(a)	$\int \sin 8x \cos 5x \, dx = \frac{1}{2}\int \sin 13x + \sin 3x \, dx$ $= -\frac{1}{26}\cos 13x - \frac{1}{6}\cos 3x + C$	**2**	2Mk: Provides correct solution. 1Mk: Correctly applies the product to sum formula.
12(b)	$V = \pi \int_0^{\ln 4} x^2 dy$ $= \pi \int_0^{\ln 4} \left(e^y\right)^2 dy$ $= \pi \left[\frac{1}{2}e^{2y}\right]_0^{\ln 4}$ $= \pi(e^{2\ln 4} - e^0)$ $= \frac{15\pi}{2} \text{u}^3$	**3**	3Mk: Provides correct solution. 2Mk: Correctly integrates e^{2y} with a correct expression. 1Mk: Obtains the correct integrand to find volume.
12(c)(i)	$A\cos(x-\alpha) = A\cos x \cos\alpha + A\sin x \sin\alpha$ $A\cos\alpha = 6$ $A\sin\alpha = 8$ $A^2 = 6^2 + 8^2$ $A^2 = 100$ $A = 10$ $\tan\alpha = \frac{8}{6}$ $\alpha = \tan^{-1}\left(\frac{8}{6}\right)$ $\therefore 8\cos x + 6\sin x = 10\cos\left(x - \tan^{-1}\frac{8}{6}\right)$	**2**	2Mk: Provides correct solution. 1Mk: Obtains A or α.
12(c)(ii)	$6\cos x + 8\sin x = 5$ $10\cos\left(x - \tan^{-1}\frac{8}{6}\right) = 5$ $\cos\left(x - \tan^{-1}\frac{8}{6}\right) = \frac{1}{2}$ $x - \tan^{-1}\frac{8}{6} = \frac{\pi}{3}, \frac{5\pi}{3}...$ $x = 1.97449..., 6.16328...$ $x = 1.974, 6.163$	**2**	2Mk: Provides correct solution. 1Mk: Obtains one correct value of x.

12(d)	 $\frac{d\theta}{dt} = ?$, when $x = 10$. And $\frac{dx}{dt} = 1.1\text{m/s}$. $\tan(\theta) = \frac{12}{x}$ $\theta = \tan^{-1}\left(\frac{12}{x}\right)$ $\frac{d\theta}{dx} = -\frac{12}{x^2+144}$ $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$ $= -\frac{12}{x^2+144} \times 1.1$ $= -\frac{12}{(10)^2+144} \times 1.1$ $= -\frac{33}{610}$ or -0.05409836... $= -0.054$ radians/second (to 3 d.p.) The angle is decreasing at a rate of 0.054 radians/second when Tom is 10 metres away.	**3**	3Mk: Provides correct solution. 2Mk: Obtains $\frac{d\theta}{dx}$ or $\frac{dx}{d\theta}$. 1Mk: Obtains θ in terms of x or x in terms of θ.
12(e)	$P(x) = (x+1)(x-3)Q(x) + ax + b$ $P(-1) = -7$ $-7 = -a + b \quad (1)$ $P(3) = 1$ $1 = 3a + b \quad (2)$ $(1) - (2)$: $-8 = -4a$ $a = 2$ $b = -7 + 2$ $= -5$ $\therefore R(x) = 2x - 5$	**3**	3Mk: Provides correct solution. 2Mk: Obtains the two simultaneous equations. 1Mk: Recognises *R(x)* is linear.

13(a)	$u=2x+1$ $du=2dx$ and $x=\frac{u-1}{2}$ when $x=4,\ u=9$ and $x=0,\ u=1$ $\int_0^4 \frac{x}{\sqrt{2x+1}}dx=\frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}du$ $=\frac{1}{4}\int_1^9 u^{\frac{1}{2}}-u^{-\frac{1}{2}}du$ $=\frac{1}{4}\left[\frac{2u^{\frac{3}{2}}}{3}-2u^{\frac{1}{2}}\right]_1^9$ $=\frac{1}{4}\left[\frac{2}{3}\left(9^{\frac{3}{2}}\right)-2\left(9^{\frac{1}{2}}\right)\right]-\frac{1}{4}\left[\frac{2}{3}\left(1^{\frac{3}{2}}\right)-2\left(1^{\frac{1}{2}}\right)\right]$ $=\frac{10}{3}$	**3**	3Mk: Provides correct solution. 2Mk: Correctly integrates the expression. 1Mk: Correctly obtains the integrand in *u*.
13(b)	Prove true for $n=1$: $\text{LHS}=\frac{1}{1\times 2}=\frac{1}{2}$ $\text{RHS}=\frac{1}{1+1}=\frac{1}{2}$ $=\text{LHS}$ The statement is true for $n=1$. Assume true for $n=k$: where $k\in\mathbb{Z}^+$ *i.e.* $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k\times(k+1)}=\frac{k}{k+1}$ Prove true for $n=k+1$: i.e. $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k\times(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$ $\text{LHS}=\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{k\times(k+1)}+\frac{1}{(k+1)(k+2)}$ $=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$ (by assumption) $=\frac{k(k+2)+1}{(k+1)(k+2)}$ $=\frac{k^2+2k+1)}{(k+1)(k+2)}$ $=\frac{(k+1)(k+1)}{(k+1)(k+2)}$ $=\frac{k+1}{k+2}$ $=\text{RHS}$	**3**	3Mk: Provides correct solution. 2Mk: Successfully does 3 of 4 key parts below. 1Mk: Successfully does 2 of 4 key parts below. Key parts: 1. Proves true for $n=1$. 2. Clearly states assumption and what must be proven. 3. Uses assumption. 4. Correctly proves required statement.

	The statement is true for $n=k+1$ if true for $n=k$. Hence if it is true for $n=1$, then it is true for $n=2,3$ and so on. $\therefore$ By the principle of mathematical induction, the statement is true for all positive integers *n*.		
13(c)(i)	$P(3\,\text{correct and 2 incorrect})=\binom{5}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$ $=\frac{45}{512}$	**2**	2Mk: Provides correct solution. 1Mk: Obtains $\binom{5}{3}\left(\frac{1}{4}\right)^3$ or $\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$.
13(c)(ii)	$P(\text{at least one incorrect})=1-P(\text{all correct})$ $=1-\binom{5}{5}\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0$ $=\frac{1023}{1024}$	**1**	1Mk: Provides correct solution.
13(d)	$E\left(\hat{p}\right)=0.07$ $\sigma\left(\hat{p}\right)=\sqrt{\frac{0.07(1-0.07)}{50}}$ $=0.0360832...$ *z*-score of 5%: $=\frac{0.05-0.07}{0.0360832}$ $=-0.55427...$ $=-0.55$ *z*-score of 10%: $=\frac{0.1-0.07}{0.0360832}$ $=0.831411...$ $=0.83$ $P(-0.55<z<0.83)=0.7967-(1-0.7088)\,(\text{from table})$ $=0.5055$ $=51\%$	**3**	3Mk: Provides correct solution. 2Mk: Obtains correct *z*-scores. 1Mk: Obtains correct standard deviation.
13(e)	$\overrightarrow{OQ}=\lambda\overrightarrow{OB}$ $\overrightarrow{OP}=\frac{2}{3}\underset{\sim}{a},\ \overrightarrow{OQ}=\lambda(\underset{\sim}{a}+\underset{\sim}{c}),\overrightarrow{PQ}=\mu\left(\underset{\sim}{c}-\frac{2}{3}\underset{\sim}{a}\right)$ $\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}$ $\therefore\lambda(\underset{\sim}{a}+\underset{\sim}{c})=\frac{2}{3}\underset{\sim}{a}+\mu\left(\underset{\sim}{c}-\frac{2}{3}\underset{\sim}{a}\right)$ $\lambda\underset{\sim}{a}+\lambda\underset{\sim}{c}=\frac{2}{3}\underset{\sim}{a}+\mu\underset{\sim}{c}-\frac{2}{3}\mu\underset{\sim}{a}$ $\left(\lambda-\frac{2}{3}+\frac{2}{3}\mu\right)\underset{\sim}{a}=(\mu-\lambda)\underset{\sim}{c}$	**3**	3Mk: Provides correct solution. 2Mk: Obtains $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ <u>and</u> $\overrightarrow{PQ}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{c}$ and a new parameter. 1Mk: Obtains $\overrightarrow{OQ}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$ or equivalent.

	Since $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are not in the same direction. Their coefficients must be 0. i.e. $\lambda-\frac{2}{3}+\frac{2}{3}\mu=0$ and $\mu-\lambda=0$ $\therefore \mu=\lambda$ $\therefore \lambda-\frac{2}{3}+\frac{2}{3}\lambda=0$ $\frac{5}{3}\lambda=\frac{2}{3}$ $\lambda=\frac{2}{5}$		

14(a)(i)	Domain: $x\in[-2,2]$ Range: $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	**2**	2Mk: Provides domain and range. 1Mk: Obtains domain or range.
14(a)(ii)	$y=\sin^{-1}\left(\frac{-x}{2}\right)$ $\sin y=\frac{-x}{2}$ $-\sin y=\frac{x}{2}$ $\sin(-y)=\frac{x}{2}$ $-y=\sin^{-1}\left(\frac{x}{2}\right)$ $y=-\sin^{-1}\left(\frac{x}{2}\right)$ Hence $f(x)$ is an odd function. [Note: Can't prove $\sin^{-1}\left(\frac{x}{2}\right)$ is odd by stating $\sin^{-1}(x)$ is odd.]	**1**	1Mk: Provides correct solution.
14(a)(iii)		**2**	2Mk: Provides correct sketch. 1Mk: Obtains correct shape or asymptote.

14(b)	$\frac{dP}{dt}=0.05P\left(\frac{C-P}{C}\right)$ $\int \frac{C}{P(C-P)}dP=\int 0.05\,dt$ $20\int \frac{1}{P}+\frac{1}{C-P}dP=\int 1dt$ $t=20[\ln P-\ln(C-P)]+D$ $=20\ln\left(\frac{P}{C-P}\right)+D$ When t = 0, P = 50 000 $0=20\ln\left(\frac{50000}{C-50000}\right)+D$ $D=-20\ln\left(\frac{50000}{C-50000}\right)$ $D=20\ln\left(\frac{C-50000}{50000}\right)$ When t = 23, P = 80 000 $23=20\ln\left(\frac{80000}{C-80000}\right)+20\ln\left(\frac{C-50000}{50000}\right)$ $23=20\left(\ln\left(\frac{80000}{C-80000}\right)+\ln\left(\frac{C-50000}{50000}\right)\right)$ $\frac{23}{20}=\ln\left(\frac{80000}{C-80000}\times\frac{C-50000}{50000}\right)$ $e^{\frac{23}{20}}=\frac{8(C-50000)}{5(C-80000)}$ $5(C-80000)e^{\frac{23}{20}}=8(C-50000)$ $5e^{\frac{23}{20}}C-400000e^{\frac{23}{20}}=8C-400000$ $\left(5e^{\frac{23}{20}}-8\right)C=400000e^{\frac{23}{20}}-400000$ $C=\frac{400000e^{\frac{23}{20}}-400000}{5e^{\frac{23}{20}}-8}$ $=110804.9149$ $=110805$ (nearest whole number) Carrying capacity is 111 000 to the nearest thousand.	**4**	4Mk: Provides correct solution. 3Mk: Obtains an expression of the constant in terms of *C*. 2Mk: Correctly integrates both sides. 1Mk: Correctly applies partial fractions.

14(c)(i)	$x = Vt\cos\theta$ ① $y = Vt\sin\theta - \frac{1}{2}gt^2$ ② From equation ① $t = \frac{x}{V\cos\theta}$ substitute into equation ② $y = V \times \frac{x}{V\cos\theta} \times \sin\theta - \frac{1}{2}g \times \frac{x^2}{V^2\cos^2\theta}$ $= x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$ Given $h = \frac{V^2}{2g}$ and $\sec^2\theta = (1 + \tan^2\theta)$ then $y = x\tan\theta - \frac{1}{4h}x^2(1 + \tan^2\theta)$	**2**	2Mk: Provides correct proof. 1Mk: Substitutes $t = \frac{x}{V\cos\theta}$ into y.
14(c)(ii)	Now (a, b) satisfies the equation $y = x\tan\theta - \frac{1}{4h}x^2(1 + \tan^2\theta)$ $b = a\tan\theta - \frac{1}{4h}a^2(1 + \tan^2\theta)$ $4hb = 4ha\tan\theta - a^2(1 + \tan^2\theta)$ $(1 + \tan^2\theta)a^2 - 4ha\tan\theta + 4hb = 0$ $a^2\tan^2\theta - 4ha\tan\theta + 4hb + a^2 = 0$ Quadratic equation has 2 solutions if $\Delta > 0$ $b^2 - 4ac > 0$ $(-4ha)^2 - 4a^2(4hb + a^2) > 0$ $16h^2a^2 - 16a^2hb - 4a^4 > 0$ $4a^2(4h^2 - 4hb - a^2) > 0$ $4h^2 - 4hb - a^2 > 0$ $a^2 < 4h^2 - 4hb$ $a^2 < 4h(h - b)$	**2**	2Mk: Provides correct solution. 1Mk: Substitutes (a,b) into equation of flight and simplifies.
14(c)(iii)	Let $\tan\theta_1$, $\tan\theta_2$ be the roots of the quadratic equation $a^2\tan^2\theta - 4ha\tan\theta + 4hb + a^2 = 0$ (from ii). By the product of the roots, $\tan\theta_1\tan\theta_2 = \frac{4hb + a^2}{a^2}$ $= 1 + \frac{4hb}{a^2}$ > 1 (since h, b, $a^2 > 0$) If $0 < \theta_1 < 45°$, then $0 < \tan\theta_1 < 1$ $0 < \theta_2 < 45°$, then $0 < \tan\theta_2 < 1$ $\therefore \tan\theta_1\tan\theta_2 < 1$, which contradicts $\tan\theta_1\tan\theta_2 = 1 + \frac{4hb}{a^2} > 1$. $\therefore$ The two balls will not pass through (a,b), given $\theta_1 < 45°$ and $\theta_2 < 45°$.	**2**	2Mk: Provides correct solution. 1Mk: Obtains sum of the roots.
	End of Paper		